$$\therefore \triangle=16\int_0^{\frac{1}{4}\pi}\int_0^{b'}\int_0^{b\sec\theta}\sqrt{(r^2-\rho^2)}\rho d\theta db d\rho \Big/ \int_0^{b'} db$$

$$=\frac{16\sqrt{2}}{3r}\int_0^{\frac{1}{4}\pi}\int_0^{b'}[r^3-(r^2-b^2\sec^2\theta)^{\frac{3}{2}}]d\theta db$$

$$=\frac{r^3\sqrt{2}}{3}\int_0^{\frac{1}{4}\pi}\left[8\sqrt{2}-(5-\sec^2\theta)\sqrt{(2-\sec^2\theta)}-6\cos\theta\sin^{-1}\left(\frac{\sec\theta}{\sqrt{2}}\right)\right]d\theta$$

$$=\frac{r^3\sqrt{2}}{3}\left[2\pi\sqrt{2}-\tfrac{3}{2}\pi\sqrt{2}-\int_0^{\frac{1}{4}\pi}\left((5-\sec^2\theta)\sqrt{(2-\sec^2\theta)}-\frac{6\tan^2\theta}{\sqrt{(2-\sec^2\theta)}}\right)d\theta\right].$$

Let $\tan\theta=\sin\phi$.

$$\therefore \triangle=\frac{r^3\sqrt{2}}{3}\left[\tfrac{1}{2}\pi\sqrt{2}-\int_0^{\frac{1}{2}\pi}\frac{4\cos^2\phi-\sin^2\phi\cos^2\phi-6\sin^2\phi}{1+\sin^2\phi}d\phi\right]$$

$$=\frac{r^3\sqrt{2}}{3}\left[\tfrac{1}{2}\pi\sqrt{2}-\int_0^{\frac{1}{2}\pi}\left(\sin^2\phi-12+\frac{16}{1+\sin^2\phi}\right)d\phi\right]=\frac{\pi r^3}{12}(23\sqrt{2}-28).$$

Also solved by the Proposer.

GROUP THEORY.

2. Proposed by W. BURNSIDE, The Croft, Bromley Road, Catford, England.

Show that the group of the biquadratic equation $x^4+2ax^2+b=0$, in which a and b are rational numbers, while a^2-b is not the square of a rational number, is in general a dihedron group of order 8; but that (*i*) if b is the square of a rational number the group is a non-cyclical Abelian group of order 4; and (*ii*) if $b=a^2\div(1+n^2)$, where n is a rational number, the group is a cyclical group of order 4.

Solution by L. E. DICKSON, Ph. D., The University of Chicago.

The problem may be solved most readily by making use of the simple expressions resulting for the roots of this special quartic. To obtain a more instructive example of the application of groups to equations, I treat the problem without assuming the quartic to be solvable by radicals.

Consider the general quartic $x^4+ax^3+bx^2+cx+d=0$, and call its roots x_1, x_2, x_3, x_4. As is well known, the functions

$$y_1=x_1x_2+x_3x_4,\quad y_2=x_1x_3+x_2x_4,\quad y_3=x_1x_4+x_2x_3$$

are the roots of the cubic resolvent

$$y^3-by^2+(ac-4d)y-a^2d+4bd-c^2=0.$$

For our special quartic, this resolvent becomes

$$y^3-2ay^2-4by+8ab\equiv(y-2a)(y^2-4b)=0.$$

We may therefore, by fixing the notation of the x's, set $y_1=2a$, $y_2=2\sqrt{b}$, $y_3=-2\sqrt{b}$. These three values are distinct since a^2-b is not the square of a rational number by hypothesis. Further, y_1 has a rational value. Hence* the group G of the quartic is contained in the dihedron group G_8,

$$I,\quad (12),\quad (34),\quad (12)(34),\quad (13)(24),\quad (14)(23),\quad (1324),\quad (1423),$$

which leaves $x_1x_2+x_3x_4$ formally unaltered. The only subgroups of order 4 of G_8 are the cyclic C_4, generated by (1324), and

$$G_4=[I,\ (12),\ (34),\ (12)(34)],\quad D_4=[I,\ (12)(34),\ (13)(24),\ (14)(23)].$$

Now the desired group G is contained in D_4 if and only if b is the square of a rational number. Indeed, $x_1x_3+x_2x_4=2\sqrt{b}$, from above, and $x_1x_3.x_2x_4=b$, from the quartic, so that $x_1x_3=x_2x_4=\sqrt{b}$. Since x_1x_3 is thus numerically unaltered by the substitions of D_4, it follows that G does not lie in D_4 if $\sqrt{b}$ is irrational. That G lies in D_4 if $\sqrt{b}$ is rational follows since the rational relation $x_1x_3=\sqrt{b}$ does not remain true upon applying (12), (34), (1324), or (1423), the quartic having no double root.

Next, G is not contained in G_4, which leaves x_1x_2 formally invariant. Indeed, $x_1x_2+x_3x_4=2a$ and $x_1x_2.\ x_3x_4=b$ give $x_1x_2=a\pm\sqrt{(a^2-b)}$, which is irrational.

Finally, G is contained in C_4 if and only if $(a^2-b)/b$ is the square of a rational number n, whence $b=a^2/(4n^2)$. The proof is simplified by the remark that the quartic contains only even powers of x so that two of its roots are the negatives of two others. In view of the values of the y_i, we may set $x_2=-x_1$, $x_4=-x_3$. Then $x_1^2+x_3^2=-y_1=-2a$. But $x_1x_3=\sqrt{b}$. Hence $x_1^2-x_3^2=2\sqrt{(a^2-b)}$. Then

$$f\equiv\frac{x_1}{x_3}-\frac{x_3}{x_1}=2\sqrt{\frac{a^2-b}{b}}.$$

As above, f is rational only when $b=a^2/(1+n^2)$. Now f is numerically unaltered by (1324), but changed into its negative by (12), (34), (13)(24), or (14)(23).

To insure that G shall not reduce to a group of order 2, necessarily generated by (12)(34), (13)(24), or (14)(23), we add a condition to insure that the quartic shall be irreducible so that its group will be transitive. Since a^2-b is not the square of a rational number, the quartic $(x^2+a)^2=a^2-b$ has no linear factor. It will have the factors $x^2\pm cx+d$ if and only if $d=\pm\sqrt{b}$, $c^2=\pm2\sqrt{b}-2a\equiv(x_1\pm x_3)^2$. We thus assume that neither of the expressions $\pm2\sqrt{b}-2a$ is the square of a rational number.

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*Dickson's *Introduction to the Theory of Algebraic Equations* (Wiley & Sons), p. 56.